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LETTER TO THE EDITOR

The Lie algebra of infinitesimal symmetries of nonlinear diffusion equations

Paul H M Kersten and Peter K H Gragert

Twente University of Technology, Department of Applied Mathematics, PO Box 217, 7500 AE Enschede, The Netherlands

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Abstract. By using developed software for solving overdetermined systems of partial differential equations in Reduce, we establish the complete Lie algebra of infinitesimal symmetries of nonlinear diffusion equations.

Using the developed software of Gragert (1981) and Gragert *et al* (1983) to do differential geometric computations on the computer and of Kersten and Gragert (1983) to solve overdetermined systems of partial differential equations, we establish the complete Lie algebras of symmetries of nonlinear diffusion equations

$$\Delta(u^{p+1}) + ku^q = u_t, \tag{1}$$

where $u = u(x, y, z, t)$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, $p, k, q \in \mathbb{R}$ $(p \neq -1)$, by application of this software to the systematic approach of Harison and Estabrook (1971) for determining the symmetries of partial differential equations.

The infinitesimal symmetries of a closed ideal I in n-dimensional space, $\mathbb{R}^n$, generated by a set of differential forms

$$\alpha(1), \ldots, \alpha(m), \tag{2}$$

are vector fields

$$V = V^i\, \partial/\partial x^i \qquad \text{(summation convention)} \tag{3}$$

such that

$$\mathscr{L}_V I \subset I, \tag{4}$$

where $\mathscr{L}_V$ denotes the Lie derivative by the vector field V (Harrison and Estabrook 1971).

From (4), we see that V has to satisfy

$$\mathscr{L}_V \alpha(i) + \gamma(i, j) \wedge \alpha(j) = 0 \qquad (i := 1 : m) \tag{5}$$

where $\gamma(i, j)$ are suitable differential forms and $\gamma(i, j) \wedge \alpha(j)$ denotes the wedge product of $\gamma(i, j)$ and $\alpha(j)$ summed over j.

Let I be the differential ideal of differential forms in

$$R^{18} = \{(x, y, z, t, u, u_x, u_y, u_z, u_t, u_{xx}, u_{xy}, u_{xz}, u_{xt}, u_{yy}, u_{yz}, u_{yt}, u_{zt}, u_{tt})\}$$

generated by

$$\begin{aligned}
\alpha(1) &= \mathrm{d}u - u_x\,\mathrm{d}x - u_y\,\mathrm{d}y - u_x\,\mathrm{d}z - u_t\,\mathrm{d}t,\\
\alpha(2) &= \mathrm{d}u_x - u_{xx}\,\mathrm{d}x - u_{xy}\,\mathrm{d}y - u_{xz}\,\mathrm{d}z - u_{xt}\,\mathrm{d}t,\\
\alpha(3) &= \mathrm{d}y_y - u_{xy}\,\mathrm{d}x - u_{yy}\,\mathrm{d}y - u_{yz}\,\mathrm{d}z - u_{yt}\,\mathrm{d}t,\\
\alpha(4) &= \mathrm{d}u_z - u_{xz}\,\mathrm{d}z - u_{yz}\,\mathrm{d}y - u_{zz}\,\mathrm{d}z - u_{zt}\,\mathrm{d}t,\\
\alpha(5) &= \mathrm{d}u_t - u_{xt}\,\mathrm{d}x - u_{yt}\,\mathrm{d}y - u_{zt}\,\mathrm{d}z - u_{tt}\,\mathrm{d}t,
\end{aligned} \tag{6}$$

where

$$u_{zz} = -u_{xx} - u_{yy} - pu^{-1}(+u_x^2 + u_y^2 + u_z^2) + (p+1)^{-1}u^{-p}(u_t - ku^q),$$

resulting from (1).

Now, application of the developed software (Gragert 1981, Gragert *et al* 1983) to (5), (6) leads to an overdetermined system of 56 partial differential equations for the components V^i of the vector field V (3).

Solving this system of partial differential equations (Kersten and Gragert 1983) leads to the final results. By inspection of the intermediate results it turns out that we have to distinguish ten cases leading to nine different Lie algebras of infinitesimal symmetries, not taking into consideration $k \neq 0$, $q = 0$. The results are a generalisation of those obtained by Branson and Steeb (1983) for dimension $N = 3$.

Because the computations are rather lengthy we shall only state the final results in the following theorems. Details of the computation can be found in Kersten and Gragert (1983).

Lemma. The nonlinear diffusion equation (1) does not admit more general Lie-contact symmetries than Lie-point symmetries (cf Anderson and Ibragimov (1979)).

Theorem 1. For any value of p, k, q the nonlinear diffusion equation (1) admits the following seven infinitesimal symmetries, i.e.

$$\begin{aligned}
&X_1 = \partial_x, \qquad X_2 = \partial_y, \qquad X_3 = \partial_z, \qquad X_4 = \partial_t,\\
&X_5 = x\partial y - y\partial_x, \qquad X_6 = x\partial_z - z\partial_x, \qquad X_7 = y\partial z - z\partial_y.
\end{aligned}$$

Theorem 2 ($p = 0$, $k = 0$). The complete Lie algebra of infinitesimal symmetries for

$$\Delta(u) = u_t$$

is spanned by 14 generators

$$X_8 = u\partial_u, \qquad X_9 = 2t\partial_z - zu\partial_u, \qquad X_{10} = 2t\partial_y - yu\partial_u, \qquad X_{11} = 2t\partial_z - zu\partial_u,$$

$$\begin{aligned}
&X_{12} = x\partial_x + y\partial_y + z\partial_z + 2t\partial_t,\\
&X_{13} = xt\partial_x + yt\partial_y + zt\partial_z + t^2\partial_t + \tfrac{1}{4}u(-x^2 - y^2 - z^2 - 6t)\partial_u,\\
&X_{14} = F(x, y, z, t)\partial_u, \text{ satisfying the PDE } \Delta(F) = F_t.
\end{aligned}$$

Theorem 3 ($p = 0$, $k \neq 0$, $q = 1$). The complete Lie algebra of infinitesimal symmetries for

$$\Delta(u) + ku = u_t$$

is spanned by 14 generators, i.e.

$$X_8 = u\partial_u, \qquad X_9 = 2t\partial_z - zu\partial_u, \qquad X_{10} = 2t\partial_y - yu\partial_u, \qquad X_{11} = 2t\partial_z - zu\partial_u,$$

$$X_{12} = x\partial_x + y\partial_y + z\partial_z + 2t\partial_t + 2kut\partial_u,$$

$$X_{13} = xt\partial_x + yt\partial_y + zt\partial_z + t^2\partial_t + \tfrac{1}{4}u(4kt^2 - x^2 - y^2 - z^2 - 6t)\partial_u,$$

$$X_{14} = F(x, y, z, t)\partial_u \qquad (\Delta(F) + kF = F_t).$$

Theorem 4 ($p = 0$, $k \neq 0$, $q \neq 1$). The complete Lie algebra of infinitesimal symmetries for

$$\Delta(u) + ku^q = u_t$$

is spanned by 8 generators, i.e.

$$X_8 = x\partial_x + y\partial_y + z\partial_z + 2t\partial_t - [2/(q-1)]u\partial_u.$$

Theorem 5 ($p = -\frac{4}{5}$, $k = 0$). The complete Lie algebra of infinitesimal symmetries for

$$\Delta(u^{1/5}) = u_t$$

is spanned by 12 generators, i.e.

$$X_8 = 4t\partial_t + 5u\partial_u, \qquad X_9 = 2x\partial_x + 2y\partial_y + 2z\partial_z - 5u\partial_u,$$

$$X_{10} = (x^2 - y^2 - z^2)\partial_x + 2xy\partial_y + 2xz\partial_z - 5xu\partial_u,$$

$$X_{11} = 2xy\partial_x + (-x^2 + y^2 - z^2)\partial_y + 2yz\partial_z - 5yu\partial_u,$$

$$X_{12} = 2xz\partial_x + 2yz\partial_y + (z^2 - x^2 - y^2)\partial_z - 5zu\partial_u.$$

Theorem 6 ($p \neq -\frac{4}{5}$, $p \neq 0$, $k = 0$). The complete Lie algebra of infinitesimal symmetries for

$$\Delta(u^{p+1}) = u_t$$

is spanned by 9 generators, i.e.

$$X_8 = -pt\partial_t + u\partial_u, \qquad X_9 = px\partial_x + py\partial_y + pz\partial_z + 2u\partial_u.$$

Theorem 7 ($p = -\frac{4}{5}$, $k \neq 0$, $q = 1$). The complete Lie algebra of infinitesimal symmetries for

$$\Delta(u^{1/5}) + ku = u_t$$

is spanned by 12 generators, i.e.

$$X_8 = \mathrm{e}^{4kt/5}\partial_t + ku\mathrm{e}^{4kt/5}\partial_u, \qquad X_9 = 2x\partial_x + 2y\partial_y + 2z\partial_z - 5u\partial_u,$$

$$X_{10} = (x^2 - y^2 - z^2)\partial_x + 2xy\partial_y + 2xz\partial_z - 5xu\partial_u,$$

$$X_{11} = 2xy\partial_x + (-x^2 + y^2 - z^2)\partial_y + 2yz\partial_z - 5yu\partial_u,$$

$$X_{12} = 2xz\partial_x + 2yz\partial_y + (z^2 - x^2 - y^2)\partial_z - 5zu\partial_u.$$

Theorem 8 ($p \neq -\frac{4}{5}$, $p \neq 0$, $k \neq 0$, $q = 1$). The complete Lie algebra of infinitesimal symmetries for

$$\Delta(u^{p+1}) + ku = u_t$$

is spanned by 9 generators, i.e.

$$X_8 = e^{-pkt}(\partial_t + ku\partial_u), \qquad X_9 = px\partial_x + py\partial_y + pz\partial_z + 2u\partial_u.$$

Theorem 9 ($p \neq 0, q = p+1$). The complete Lie algebra of infinitesimal symmetries for

$$\Delta(u^{p+1}) + ku^{p+1} = u_t$$

is spanned by 8 generators, i.e.

$$X_8 = pt\partial_t - u\partial_u.$$

Theorem 10 ($p \neq 0, p \neq -\frac{4}{5}, q \neq 1, q \neq p+1$). The complete Lie algebra of infinitesimal symmetries for

$$\Delta(u^{p+1}) + ku^q = u_t$$

is spanned by 8 generators, i.e.

$$X_8 = (-p+q-1)(x\partial_x + y\partial_y + z\partial_z) + 2(q-1)t\partial_t - 2u\partial_u.$$

By the use of symbolic computations, we derived the complete Lie algebra of infinitesimal symmetries for nonlinear diffusion equations, generalising the results of Branson and Steeb (1983) in the case $N = 3$, and proved that there are no other Lie-contact symmetries than Lie-point symmetries.

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